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Symmetries of differential equations

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Abstract. The use of a symmetry to reduce the order of an nth-order differential equation is treated by considering the symmetry of an associated vector field. A particular choice of associated vector field leads to the usual extension of the Lie symmetry method. The possibility of other choices leads to a powerful generalisation. An algebraic classification of transformations arises naturally from the theory. It is shown to be equivalent to the geometric classification of transformations as contact and non-contact.

1. Introduction

Sophus Lie introduced the method of reduction of a differential equation through the use of a one-parameter invariance group [1]. A number of authors have generalised Lie's method and applied it to a variety of problems [2, 3]. Recent work has been prompted by the increasing prominence of non-linear problems and the need to develop techniques to tackle them. The work of Lie, as well as all subsequent generalisation, has been based on the concept of the extended transformation, which is the operation that extends the action of the one-parameter group from the two-dimensional space (x, y) to an n-dimensional space $(x, y, y', \ldots, y^{(n-2)})$ where n will depend on the order of the differential equation considered. A good exposition of this technique can be found in Bluman and Cole [3].

In this paper we present an alternative approach that has the advantage of being both simpler and more general than the method of the extended group. This approach is formulated in two steps. First, in § 2, we consider the condition for a vector field (or dynamical system) to be invariant under a one-parameter group of transformations and show how this vector field is reduced to a 'simpler' vector field [4].

We then use the well known result that a differential equation can be represented as a dynamical system and obtain the reduction of the differential equation as a direct consequence of the reduction of the corresponding dynamical system. The representation of the differential equation as a dynamical system is not unique. In the case of a second-order differential equation it depends on an arbitrary function of three variables $Y(x, y, z)$. A particular choice of Y, namely $Y(x, y, z) = z$, leads to the previous generalisation of Lie's work [5]. The generalisation presented here corresponds to choosing any other value for Y.

One of the consequences of this generalisation is the possibility of always dealing with a one-parameter group of contact transformations. In the previous formulations, although most authors have restricted themselves to point and contact transformations,

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it is possible for a one-parameter group of transformations to be neither point nor contact. We call such transformations non-contact transformations. It is usually harder to use a non-contact than a contact group for the reduction of order of a differential equation. However, we show in § 3 that it is always possible to choose $Y(x, y, z)$ such that the one-parameter group considered is a one-parameter group of contact transformations. We show in the appendix that our notions of point, contact and non-contact transformations, though introduced according to purely algebraic criteria, are equivalent to those introduced in modern differential geometry [6] according to purely geometric criteria and go beyond Lie's original definition [1]. In § 4 we give an example of the method and summarise our results.

2. Dynamical systems and second-order equations

Consider the n-dimensional autonomous dynamical system

$$\mathrm{d}x_i/\mathrm{d}t = V_i(x_1, \ldots, x_n) \qquad i = 1, \ldots, n \tag{1}$$

corresponding to the vector field $V = [V_i \partial/\partial x_i]$. The dynamical system is said to be invariant under a group of one-parameter transformations generated by the vector field $V = V_i\partial/\partial x_i$ if the Lie bracket (commutator) of U and V vanishes:

$$[U, V] = 0. \tag{2}$$

When U is expressed in its comoving coordinates $u_1, \ldots, u_n$ it becomes a unit vector field, say in the direction u_1 [8],

$$U = \partial/\partial u_1 \tag{3}$$

and since (2) is a coordinate-independent relation it becomes

$$[\partial/\partial u_1, V] = (\partial_{u1} V_i(u_1, \ldots, u_n))\partial/\partial u_i = 0 \tag{4}$$

where $[V_i]$ are the components of V in the $[u_i]$ coordinates. From (4) one concludes that all the V_i are independent of u_1. The dynamical system in these coordinates is

$$\mathrm{d}u_i/\mathrm{d}t = V_i(u_2, \ldots, u_n) \qquad i = 1, \ldots, n \tag{5}$$

which is an $(n-1)$-dimensional system plus a quadrature for u_1. A weaker form of invariance of V under U is obtained if

$$[U, V] = \lambda V \tag{6}$$

where λ is an arbitrary function. In the comoving coordinates of V equation (6) becomes

$$\partial V_i/\partial u_1 = \lambda V_i \tag{7}$$

which implies that the V_i are of the form

$$V_i = C(u_1, \ldots, u_n)R_i(u_2, \ldots, u_n) \qquad i = 1, \ldots, n \tag{8}$$

where the function C is common to all the components. Now only the ratio of any two components of V will be independent of u_1. However, in representing differential equations by vector fields, the t parametrisation of the corresponding dynamical systems will be unimportant. The important object is the path traced out in the n-dimensional space by a solution of the dynamical system. This is given by the characteristic system

$$\mathrm{d}u_1/V_1 = \mathrm{d}u_2/V_2 = \ldots = \mathrm{d}u_n/V_n \tag{9}$$

which is independent of the t parametrisation, and depends only on the ratios of the components of V. Hence the weaker invariance (6) will be sufficient for our purpose.

Now consider any second-order differential equation of the general form

$$\mathrm{d}^2y/\mathrm{d}x^2 = \omega(x, y, \mathrm{d}y/\mathrm{d}x) \tag{10}$$

and a corresponding dynamical system

$$\mathrm{d}x/\mathrm{d}t = 1 \qquad \mathrm{d}y/\mathrm{d}t = Y(x, y, z) \qquad \mathrm{d}z/\mathrm{d}t = Z(x, y, z) \tag{11}$$

where we have used the arbitrariness of the t parametrisation to choose the first component to be one. Dividing the first two equations of (11) gives

$$\mathrm{d}y/\mathrm{d}x = Y(x, y, z). \tag{12a}$$

Taking the derivative of (12a) yields

$$\mathrm{d}^2y/\mathrm{d}x^2 = Y_x + Y_y\,\mathrm{d}y/\mathrm{d}x + Y_z\,\mathrm{d}z/\mathrm{d}x \tag{12b}$$

where subscripts denote partial differentiation. Now equations (12) will be the original differential equation (10) if one chooses Z and Y so that

$$\omega(x, y, Y) = Y_x + Y_yY + Y_zZ. \tag{13}$$

Equations (11) and (13) are thus equivalent to (10).

We now show how the condition of invariance of a dynamical system translates to that for a differential equation.

The conditions (6) that $U = (\xi, \eta, \zeta)$ be a symmetry of $V = (1, Y, Z)$ are just

$$-(\xi_x + Y\xi_y + Z\xi_z) = \lambda \tag{14}$$

$$\xi Y_x + \eta Y_y + \zeta Y_z - \eta_x - Y\eta_y - Z\eta_z = \lambda Y \tag{15}$$

$$\xi Z_x + \eta Z_y + \zeta Z_z - \zeta_x - Y_y - Z\zeta_z = \lambda Z. \tag{16}$$

Using (14) to eliminate λ from (15) and (16) we obtain

$$\zeta = [\eta_x + Y(\eta_y - \xi_x) - \xi Y_x - \eta Y_y - \xi_y Y^2 + Z(\eta_z - Y\xi_z)]/Y_z \tag{17}$$

$$\xi Z_x + \eta Z_y + \zeta Z_z = \zeta_x + Y\zeta_y - (\xi_x + Y\xi_y - \zeta_z)Z - \xi_z Z^2. \tag{18}$$

Equations (12), (17) and (18) are therefore the most general equations giving the condition of invariance of the differential equation (10) under a one-parameter group of transformations $U = (\xi, \eta, \zeta)$.

In the case where the choice $Y(x, y, z) = z$ is made, (17) and (18) reduce to

$$\zeta = \eta_x + z(\eta_y - \xi_x) - \xi_y z^2 + (\eta_z - z\xi_z)\omega \tag{19}$$

$$\xi\omega_x + \eta\omega_y + \zeta\omega_z = \zeta_x + Z\zeta_y - (\xi_x + z\xi_y - \zeta_z)\omega - \xi_z\omega^2. \tag{20}$$

These equations were given by Meinhardt [5].

We shall now show how the reduction of order of the differential equation occurs. Having chosen a function Y and a group whose generators satisfy (17) and (18), suppose we can integrate the group to find the comoving coordinates $u_1(x, y, z)$, $u_2(x, y, z)$, $u_3(x, y, z)$. The vector field associated with the differential equation is $V = (1, Y(x, y, z), Z(x, y, z))$ in the x, y, z coordinate system and must be of the form $V_i = C(u_1)R_i(u_2, u_3)$ in the comoving coordinate system as we have seen in (8). These components are related by the usual vector transformation laws which are, for the last two components,

$$C(u_1)R_2(u_2, u_3) = \partial u_2/\partial x + Y\partial u_2/\partial y + Z\partial u_2/\partial z = \dot{u}_2 \tag{21}$$

$$C(u_1)R_3(u_2, u_3) = \partial u_3/\partial_x + Y\partial u_3/\partial y + Z\partial u_3/\partial z = \dot{u}_3. \tag{22}$$

Dividing (21) by (22) and setting $F(u_2, u_3) = R_2/R_3$ we obtain

$$F(u_2, u_3) = \frac{\partial u_2/\partial x + Y\partial u_2/\partial y + Z\partial u_2/\partial z}{\partial u_3/\partial x + Y\partial u_3/\partial y + Z\partial u_3/\partial z} = \mathrm{d}u_2/\mathrm{d}u_3 \tag{23}$$

which is a first-order differential equation in the variables u_2 and u_3. Thus, the second-order differential equation (10) is reduced to the first-order equation (23) by virtue of the invariance under the group $U = (\xi, \eta, \zeta)$ provided, of course, the group equations can be integrated to find the comoving coordinates.

3. Point, contact and non-contact transformations

Let us first examine equations (19) and (20) which are the invariance condition for the case $Y = z$. If η and ξ are assumed then (19) and (20) are two coupled partial differential equations for the functions ζ and ω. However, these equations will decouple whenever we have

$$\eta_z - z\xi_z = 0. \tag{24}$$

This will happen trivially for $\xi_z = \eta_z = 0$, i.e. when the generators ξ and η are independent of z. This is the well known condition for a point transformation of the plane [1]. We will now show that if (24) is satisfied with $\xi_z \neq 0$ and $\eta_z \neq 0$ we do have a contact transformation, in the sense of Lie. Such transformations have been characterised by Lie [1] as having generating functions $W(x, y, z)$ such that

$$\xi(x, y, z) = -W_z \tag{25}$$

$$\eta(x, y, z) = W - zW_z \tag{26}$$

in which case (19) reduces to

$$\zeta = W_x + zW_y. \tag{27}$$

Now for ξ and η given by (25) and (26) we can see that (24) is automatically satisfied. Conversely, integrating (24) we obtain

$$\eta = z\xi - \int \xi \,\mathrm{d}z + G(x, y). \tag{28}$$

By setting

$$W(x, y, z) = -\int \xi \,\mathrm{d}z + G(x, y) \tag{29}$$

we see that (25) and (26) are satisfied thus proving the claim that (24) is the condition for a contact transformation. A group of transformations whose infinitesimal coordinate functions ξ and η do not satisfy (24) will be called a non-contact group of transformations.

Returning to the general case, we see that (17) and (18) are also two coupled partial differential equations for ζ and Z once ξ, η and Y have been chosen. Similarly (17) and (18) will decouple if we have

$$\eta_z - Y\xi_z = 0 \tag{30}$$

which is automatically satisfied for point transformations. More generally, if (30) is satisfied one can easily show by integration that there exists a 'generalised generating function' $W(x, y, z)$ such that

$$\xi = -W_z/Y_z \tag{31}$$

$$\eta = W - YW_z/Y_z \tag{32}$$

$$\zeta = (W_x + YW_y - Y_yW)/Y_z \tag{33}$$

which reduce to (25), (26) and (27) for $Y = z$, as they should. Moreover, we will show in the appendix that a group of transformations satisfying (30) coincides with the definition of a contact transformation as formulated in modern differential geometry. We are therefore justified in calling a group of transformations satisfying equation (30) a group of contact transformations. It is clear that while the fact that a group is a group of point transformations does not depend on the choice of $Y(x, y, z)$, this is not the case for contact transformations. Indeed, if $\xi_z = \eta_z = 0$ both (24) and (30) are satisfied: otherwise there is only one choice of Y, namely $Y = \eta_z/\xi_z$, that satisfies (30) once ξ and η have been chosen. Therefore, one speaks of a group of contact transformations of a specified vector field, $(1, Y, Z)$ in our case. Also, if (30) is not satisfied we will speak of a group of non-contact transformations of $(1, Y, Z)$.

A role for Y in selecting a convenient type of one-parameter group now becomes apparent. For given ξ and η it is always possible to choose Y so that (30) is satisfied, thus obtaining a contact transformation and the decoupling of equations (17) and (18). This is one way Y may be used in the inverse problem of determining the differential equations invariant under a given group. Once ξ and η are chosen and Y is chosen in order to ensure a contact transformation we then have ζ completely determined by (17). Finally, eliminating Z between (13) and (17) we obtain a unique linear partial differential equation for ω whose first two characteristic equations are identical to those necessary to solve for the group integration itself.

For the direct problem, that of searching for groups of invariance of a given differential equation, we proceed as follows. Since we are looking for contact transformations in the generalised sense, a generalised generating function exists and we have ξ, η and ζ given be (31), (32) and (33). Hence the only equation we will have to solve will be equation (18) which is now a partial differential equation for the two functions W and Y. For example, if we impose the two restrictions $Y = z$ and W linear in z (point transformation) [1, 4], we then have $Z = \omega$ and the only equation we must solve is

$$\begin{aligned} -\xi\omega_x - \eta\omega_y - [\eta_x + (\eta_y - \xi_x)z - \xi_y z^2]\omega_z + \eta_{xx} + (2\eta_{xy} - \xi_{xx})z + (\eta_{yy} - 2\xi_{xy})z^2 \\ - \xi_{yy}z^3 + (\eta_y - 2\xi_x - 3\xi_y z)\omega = 0 \end{aligned} \tag{34}$$

where ξ and η are functions of x and y. This particular equation was given by Bluman and Cole [3]. It is clear though that we obtain a different equation for each choice of Y and W.

4. Example and conclusion

In this section we give a specific example, albeit very simple, of an equation for which Y can be chosen to give a trivial integration even more simple than other more conventional choices. The equation

$$d^2y/dx^2 = (dy/dx)^2/2y - (dy/dx)/x \qquad (35)$$

is clearly invariant under the two one-parameter groups

$$x \to \alpha x$$

and

$$y \to \beta y.$$

In the theory outlined above these groups correspond to $Y = z$, $W = xz$ and $Y = z$, $W = y$ respectively. The first choice leads to $(\xi, \eta, \zeta) = (-x, 0, z)$ and to the invariants $u_2 = y$, $u_3 = xy'$. In these coordinates (35) becomes

$$du_3/du_2 = u_3/2u_2 \qquad (36)$$

which integrates to

$$u_3 = Ku_2 \qquad (37)$$

i.e.

$$(dy/dx)^2 = Ky/x^2. \qquad (38)$$

Similarly, the choice $Y = z$, $W = y$ leads to the group coordinates $(\xi, \eta, \zeta) = (0, y, z)$ and to the invariants $u_2 = x$ and $u_3 = y'/y$ and to the equation

$$du_3/du_2 = -(u_2^3/2 + u_3/u_2) \qquad (39)$$

which is non-linear. Both (38) and (39) can be integrated, for example, by finding invariances of these equations. However this is an additional reduction.

On the other hand, choosing $Y = z/x$ gives rise to the following possibilities for W: $W = y$, $W = z$ and $W = z^2$. One can show that these are the only possibilities if we allow W to be a third degree polynomial in x, y and z. Consider the choice $W = z^2$. It leads to to $(\xi, \eta, \zeta) = (-xz, -z^2, 0)$, $u_2 = xy'$ and $u_3 = \log x - y/y'x$. In these coordinates (35) reads

$$du_3/du_2 = 0 \qquad (40)$$

which integrates to

$$u_3 = K \qquad (41)$$

i.e.

$$y'/y = 2/(x \log K_1 x) \qquad (42)$$

which is in quadrature. Integration yields

$$y = K_2(\log K_1 x)^2 \qquad (43)$$

which is the solution to our equation.

In conclusion, mapping a differential equation to the most general associated vector field considerably enlarges the scope of previous theories of reduction of such equations.

The possibility of always being able to choose a contact transformation is a feature unique to the present work. To solve the direct problem for a second-order equation one needs to solve a single partial differential equation for two functions of three variables and although this equation contains many terms, it can readily be handled by a symbolic manipulation language such as REDUCE or SMP. A program has been written that finds the symmetries generated by polynomial W and rational Y for differential equations given by rational W [7]. The example given has been solved using that program.

Appendix

In modern differential geometry one defines a contact transformation as follows [6].

First define a contact 1-form by

$$\delta \wedge \mathrm{d}\delta \neq 0. \tag{A1}$$

The infinitesimal generator of a contact transformation is then a vector field X such that

$$L_X\delta = \lambda\delta \tag{A2}$$

where L_X denotes the Lie derivative and where λ is an arbitrary function.

In § 3 we have set $\mathrm{d}y/\mathrm{d}x = z$. Now consider a three-dimensional space R^3 with coordinates x, y, z and consider a 2-surface S such that the form

$$\delta = \mathrm{d}y - z\,\mathrm{d}x \tag{A3}$$

annihilates all vector fields tangent to S.

We have

$$\mathrm{d}\delta = -\mathrm{d}z \wedge \mathrm{d}x \tag{A4}$$

and

$$\delta \wedge \mathrm{d}\delta = -(\mathrm{d}y - z\,\mathrm{d}x) \wedge (\mathrm{d}z \wedge \mathrm{d}x) = -\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \neq 0.$$

δ is therefore a contact 1-form. The vector field $X = \xi\partial_x + \eta\partial_y + \zeta\partial_z$ will be the infinitesimal generator of a contact transformation if

$$L_X\delta = \gamma\delta. \tag{A2}$$

But

$$\begin{aligned} L_X\delta = i_X\mathrm{d}\delta + \mathrm{d}i_X\,\delta &= -(\xi\partial_x + \eta\partial y + \zeta\partial_z)(\mathrm{d}z \wedge \mathrm{d}x) + \mathrm{d}(-z\xi + \eta) \\ &= (\eta_x - z\xi_x - \zeta)\,\mathrm{d}x + (\eta_y - z\xi_y)\,\mathrm{d}y + (\eta_z - z\xi_z)\,\mathrm{d}z \end{aligned} \tag{A5}$$

and

$$\gamma\delta = \gamma(\mathrm{d}y - z\,\mathrm{d}x). \tag{A6}$$

Using (A2), (A5) and (A6), identifying the coefficients of $\mathrm{d}x$, $\mathrm{d}y$ and $\mathrm{d}z$ and eliminating γ we obtain

$$\eta_z - z\xi_z = 0 \tag{A7}$$

and

$$\zeta = \eta_x + z(\eta_y - \xi_x) - z^2\xi_y. \tag{A8}$$

(A7) and (A8) are equivalent to the pair of equations (19) and (24) which are the conditions for a contact transformation with $Y = z$. Therefore, in the case where the contact form is given by (A3), the modern definition of a contact transformation coincides with our definition. Clearly, it also coincides with Lie's since we showed that our definition coincides with Lie's for the case $Y = z$.

If we now consider the 1-form

$$\delta = \mathrm{d}y - Y(x, y, z)\,\mathrm{d}x \tag{A9}$$

we can carry out the same argument and calculations as above. The result is that for $X = \xi\partial_x + \eta\partial_y + \zeta\partial_z$ to be a contact transformation in the sense of modern differential geometry, we must have

$$\eta_z - Y\xi_z = 0 \tag{A10}$$

and

$$\zeta = [\eta_x + Y(\eta_y - \xi_x) - \xi Y_x - \xi Y_y - Y^2\xi_y]/Y_z. \tag{A11}$$

Equations (A10) and (A11) are equivalent to the pair (17) and (29) showing that our generalised definition of a contact transformation stemming from algebraic grounds coincides with the modern differential geometric definition and goes beyond Lie's original definition.

We have therefore justified the term contact transformation, used throughout this paper, by both exhibiting the generating function and by showing it agrees with the modern definition of such transformations.

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